Linear Stability Theory and the Problem of Supersonic Boundary-Layer Transition

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Compressible linear stability theory is first reviewed and then used to calculate the amplitude ratio of constant-frequency disturbances as a function of Reynolds number for insulated and cooled-wall flat-plate boundary layers between Mach numbers 1.3 and 5.8. These results are used to examine the consequences of using a fixed disturbance amplitude of the most unstable frequency as a transition criterion. The effect of the freestream Mach number M_1 on the transition of insulated-wall boundary layers is calculated using two different assumptions concerning the initial boundary-layer disturbance amplitude A_o. It is found that the shape of the transition Reynolds number Re_t vs M_1 curve observed in wind tunnels can be closely duplicated. As a second example, the effect of wall cooling at $M_1 = 3.0$ is calculated. A much faster increase of Re_t with cooling is obtained than is observed experimentally. However, when A_o is determined from the forced response of the boundary layer to irradiated sound and from the measured freestream power spectrum, a rise in Re_t similar to what is observed is obtained for a certain amplitude criterion.

Nomenclature

A = disturbance amplitude
A_1 = disturbance amplitude in freestream
A_o = initial disturbance amplitude in boundary layer
A_r = reference initial disturbance amplitude in boundary layer at $M_r (= 1.3)$
c = $c_r + ic_i$, complex wave velocity $(= \omega/\alpha)$
c_p = ω/α_r, phase velocity
F = ω/R, dimensionless frequency
L^* = $(x^* \nu_1^* / U_1^*)^{1/2}$, Blasius length scale
m = rms amplitude of mass-flow fluctuation
m_I = m of incoming wave
m_p = peak value of m in boundary layer
M = Mach number
$\tilde{M}$ = $M_1(U - c_r)/T^{1/2}$, Mach number of relative flow
Re = $U_1^* x^* / \nu_1^*$, x-Reynolds number based on edge conditions
R = $Re^{1/2}$, Reynolds number based on Blasius length scale
Re_A = Re where disturbance amplitude is A
Re_t = Re at transition
t = time
T = temperature
U = mean velocity in stream direction
x, y, z = streamwise, normal, and lateral coordinates
α_r = wave number in stream direction
α_i = spatial amplification rate
β = complex wave number in lateral direction
γ = ratio of specific heats
δ = boundary-layer thickness ($= y^*$ at $U = 0.999$)
ν = kinematic viscosity
π = pressure-fluctuation amplitude function
ψ = $\tan^{-1}(\beta_r/\alpha_r)$, wave obliqueness angle
ω = frequency

Superscripts

$*$ = dimensional quantity
= y derivative

Subscripts

aw = adiabatic wall condition
c = cone condition
f = flat-plate condition
o = evaluated at first neutral-stability point
s = evaluated at generalized inflection point
w = wall condition
1 = boundary-layer edge condition

I. Introduction

THE idea that the origin of turbulence is to be sought in the instability of laminar flow dates back to the 19th century. The first notable advance in the application of this idea to boundary layers was made by Prandtl,[1] who demonstrated on the basis of a simple linear model the destabilizing effect of viscosity. This finding was important because it brought forward the instability mechanism in the simplest of all boundary layers, the incompressible boundary layer on a flat plate, or Blasius boundary layer. Subsequent work by Tollmien and Schlichting resulted in a highly-developed theory for the stability of the Blasius boundary layer. Later, calculations were made of the stability of a large number of boundary layers with favorable and adverse pressure gradients. A good review of this early work with an extensive list of references is to be found in Schlichting's book.[2]

The purpose of these investigations was to gain a better understanding of the occurrence of transition. However, the main result of the theory, that the disturbance frequency is of primary importance, had no experimental support until the experiments of Schubauer and Skramstad[3] showed conclusively that instability waves exist in a boundary layer with small external disturbances, and that the theory of Tollmien and Schlichting accounts for their behavior. Schubauer and Skramstad also showed that instability waves were a necessary precursor of transition at the low disturbance level of their wind tunnel, and that when sinusoidal disturbances were introduced artificially into the boundary layer, the position of transition could be varied by changing either the frequency or the amplitude of the disturbances. The importance of the spectrum of the external disturbances was demonstrated by Spangler and Wells.[4] They found a much higher transition Reynolds number on a flat plate than Schubauer and Skramstad even though the freestream disturbance amplitudes were similar in the two experiments. However, the spectra of the freestream disturbances were radically different.

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Even with large external disturbances (0.42%), Bennett[5] showed that instability waves precede transition, and Klebanoff and Tidstrom[6] showed that instability amplification is still the precursor to transition in the flow behind a two-dimensional roughness element of moderate size. Another example of frequency-sensitive transition was provided by the experiments of Jackson and Heckle.[7] Here the disturbance amplitude necessary to maintain transition at a fixed position was found to be extremely sensitive to the disturbance frequency at a freestream turbulence level of 0.2–0.4%.

In the absence of any definitive transition theory, several approaches have been tried. A first approach is to take some simple physical idea and develop a formula with adjustable constants. Liepmann,[8] many years ago, advanced the idea that transition is initiated when the ratio of the disturbance Reynolds stress to the mean viscous stress reaches some critical value and worked out a relation that involves the freestream turbulence level, unit Reynolds number, and the linear stability theory. A simpler version of this approach which does not use the stability theory was advanced by Van Driest and Blumer,[9] and more recently by Benek and High.[10] The correlation of Pate and Scheuler[11] for transition in supersonic wind tunnels, which is based on the idea that the controlling factor is the irradiated sound from the turbulent wall boundary layers, is another example of a physically based correlation.

A second approach is to use turbulence model equations to follow downstream a specified disturbance introduced into the laminar boundary layer. Calculations of this kind have been carried out by Yates and Donaldson[12] and by Wilcox.[13] One difficulty with this method is that the disturbance frequency, which has been established to be of dominant importance in transition, does not enter these theories in a direct manner. It is only by identifying a turbulent length scale with the disturbance wave length that a connection can be made at all with the large body of careful experimental results with controlled sinusoidal disturbances.

Yet a third approach is to apply the linear stability theory directly to the transition problem. This theory suffers the obvious disadvantage of being linear, but against this limitation must be weighed the great advantage that one is dealing with solutions of the Navier-Stokes equations. The only assumption other than linearization is that of parallel flow. Both the available experiments and recent theoretical work at low speed[14] show that this assumption is a good one for boundary-layer flows. One major use of the stability theory in the past has been the calculation of the initial point of instability, or critical Reynolds number, as a function of some parameter of the mean flow. Although the critical Reynolds number is at times a useful quantity in stability theory, it is too remote from transition in most instances to serve as an indication of transition dependence on the mean flow. The amplification of a disturbance is the decisive factor, not its initial point of instability. Calculations of the amplitude growth of constant-frequency disturbances led to the well known e^9 criterion of Smith and Gamberoni[15] and of Van Ingen,[16] later modified to e^{10} by Jaffe, Okamura, and Smith.[17] In the latter reference, transition was found to occur in low turbulence wind tunnels at amplitude ratios of $e^{6.8}$ to $e^{12.1}$. It is easy to criticize this approach, but its general success, particularly in giving the change in transition location with freestream Reynolds number for the same model in the same wind tunnel and using a single fixed but optimum value of the amplitude ratio, does indicate that there are cases where linear instability is the controlling factor in transition, and the nonlinear part of the transition process can be accounted for by the simple expedient of letting the disturbance grow to some arbitrary and unrealistically high level.

Boundary-layer transition at supersonic speeds is a confusing and perplexing subject. A good review of present knowledge and a discussion of the many unresolved issues have been given by Morkovin.[18] The idea of using linear stability theory as a means of clarifying some of these issues was advanced by Reshotko.[19] It is the purpose of this paper, based on Reshotko's idea and the achievements of the e^9 method in low-speed flow, to determine if detailed numerical results obtained from compressible stability theory are in accord with reliable transition measurements made in supersonic wind tunnels. The investigation will focus on two points: a) the variation of transition Reynolds number with freestream Mach number for an adiabatic flat plate, and b) the effect of wall cooling at a single fixed Mach number. These two cases offer an instructive contrast because in the first the disturbance environment is changing and in the second it is constant. The aim of the present investigation is not to advance a method for predicting the transition Reynolds number. Rather it is to determine if the observed behavior of the transition Reynolds number in the above two cases is a result of what happens in that portion of the boundary layer where the linear stability theory is valid. Strong support for this idea will be found when the stability theory is supplemented by information about the external disturbances. For the cooled-wall boundary layer, the initial amplitude of disturbances in the boundary layer is critical, and a variant of stability theory which accounts for the forced response of the boundary layer to a particular type of external sound field as well as information about the disturbance spectrum will prove to be needed to arrive at a rise in transition Reynolds number with cooling which is similar to what is observed in a wind tunnel.

II. Review of Stability Theory

For compressible flow, the linear stability theory which was developed on the basis of the asymptotic techniques of incompressible flow theory by Lees and Lin,[20] and subsequently extended by Dunn and Lin[21] and Lees and Reshotko,[22] proved to be valid only up to low supersonic Mach numbers. This approach has been supplanted by direct computer solutions of the governing differential equations, as has been the case in most stability problems in recent years. The first work in this direction was done by Brown,[23] and a different numerical method was developed independently by the present author.[24] A presentation of the theory and extensive numerical results for zero pressure-gradient boundary layers up to a freestream Mach number of ten can be found in Ref. 25. A summary of the principal conclusions which have been derived from the mass of numerical computations is given below. All of the numerical results of this paper have been computed from the parallel-flow form of linear stability theory. Only the local boundary-layer profile is considered and there are no terms involving either the normal velocity or the x-derivatives of the mean flow. Versions of the theory which introduce terms of this sort[26–28] have led so far to inferior results† as judged by the stability experiments of Kendall.[29]

The incompressible flat-plate boundary layer has no inflection point, and consequently is stable to inviscid disturbances; the most unstable disturbance is two-dimensional; and there is at most a single unstable frequency at each Reynolds number and wave angle. None of these statements is true for the compressible flat-plate boundary layer. As shown by Lees and Lin,[20] the inflection-point criterion for inviscid instability is replaced in compressible flow by $(U'/T)_s' = 0$. The location of the above zero, y_s, will be called the generalized inflection point. A generalized inflection point exists in all insulated-wall flat-plate compressible boundary layers, and there can be two such points for cooled-wall boundary layers. As the Mach number increases, y_s moves away from the wall and the inviscid instability increases. Simultaneously, the viscous instability mechanism weakens, so that when the freestream Mach number M_1 reaches about 3, the maximum amplification rate occurs as the Reynolds number approaches infinity and viscosity has only a stabilizing influence just as in a free shear flow.

† The numerical results of Brown[26] which were obtained from equations with additional terms inexplicably suffer from bad scatter. Boehman[28] redid the whole calculation with an improved numerical procedure.

Perhaps the most striking departure of supersonic boundary-layer instability from incompressible experience lies in the multiplicity of unstable modes that are present whenever there is a region of supersonic mean flow relative to the disturbance phase velocity.[30] The reason for the additional modes can be seen easily from an examination of the second-order inviscid stability equation for the pressure-fluctuation amplitude function $\pi(y)$ of a disturbance of neutral stability $\pi(y) \exp [i(\alpha x - \omega t)]$.

$$\pi'' - (\log \hat{M}^2)'\pi' - \alpha^2(1-\hat{M}^2)\pi = 0 \qquad (1)$$

In Eq. (1), $\hat{M}$ is the local Mach number of the mean flow relative to the disturbance phase velocity c_r. The quantities U, c_r, and T are all dimensionless with respect to the freestream velocity $U_1{}^*$ and temperature $T_1{}^*$. If U and α are taken in the direction normal to the constant-phase line in the x-z plane, Eq. (1) is also valid for three-dimensional disturbances $\pi(y) \exp [i(\alpha x + \beta z - \omega t)]$. When supersonic relative flow exists from the wall to y_a, Eq. (1) is a wave equation in this region and multiple standing-wave solutions in the y direction can satisfy the boundary conditions at the wall and y_a. Since each standing wave length in the y direction is nothing more than the disturbance wave length $2\pi/\alpha$ times a factor involving the mean relative flow, there must be multiple values of α. The additional α_n, called the higher modes, are all unstable, and the first additional, or second, mode is the most unstable of all the modes. The effect of viscosity on the higher modes is always stabilizing, so the maximum amplification rate occurs as the Reynolds number approaches infinity. The higher maximum inviscid amplification rate of the second mode compared to the first mode is translated, at Mach numbers above about 4, into a lower critical Reynolds number.

In contrast to incompressible flow, where a two-dimensional disturbance is the most unstable at any Reynolds number, for supersonic flow the most unstable first-mode disturbance is always oblique. The wave angle ψ of the most unstable disturbance increases rapidly with Mach number and is in the range from 55°–60° above $M_1 = 1.6$. It might be noted that as far as the first mode is concerned, the most unstable adiabatic flat-plate boundary layer occurs at $M_1 = 0$. Even the most unstable oblique inviscid disturbance (at about $M_1 = 4.5$, $\psi = 60°$) has a lower amplification rate than the maximum incompressible viscous amplification rate.

For the second and higher modes, two-dimensional disturbances are the most unstable. Even with the increased amplification of the second mode compared to the first mode, the most unstable second-mode disturbance has a spatial amplification rate less than the maximum viscous amplification rate at $M_1 = 0$. These remarks on the maximum amplification rates emphasize the comparative stability of a supersonic flat-plate boundary layer with respect to its low speed counterpart. The latter, in turn, is only weakly unstable if compared with an adverse pressure-gradient boundary layer or free shear flow.

One of the earliest and best-known results of linear stability theory for supersonic flow was the prediction by Lees[31] that complete stabilization can be achieved by wall cooling over a wide range of Mach numbers. Since the criterion for complete stabilization was based on the asymptotic theory of two-dimensional disturbances and did not take the higher modes into account, it is perhaps surprising that it has any validity at all. The calculations of Ref. 25, as well as a calculation in Sec. IV at $M_1 = 3.0$, fully support the idea that the first mode is strongly stabilized by cooling. At high Mach numbers, complete stabilization is not possible, again in accord with Lees' prediction. More cooling is required to stabilize oblique disturbances than two-dimensional disturbances, but the effect is the same. However, the higher modes change this picture in an important way. Cooling not only does not stabilize the higher modes, it destabilizes them. The higher modes depend only on the existence of a region of supersonic relative flow. Even though a generalized inflection point is necessary for neutral inviscid higher-mode disturbances, amplified higher-mode disturbances can exist at both infinite and finite Reynolds number even in the absence of a generalized inflection point. As a consequence, the usual procedures which stabilize low-speed boundary layers such as wall cooling, suction, and favorable pressure gradients cannot be expected to have a similar effect when the second mode is of importance. The frequencies of the most unstable second-mode disturbances at moderate supersonic Mach numbers under wind tunnel conditions are usually too high to coincide with energy containing frequencies of external disturbances, as pointed out by Reshotko.[19] For hypersonic flow the situation changes, and it is above $M_1 = 6$ or 7 that the second mode can be expected to be of importance, particularly for cooled-wall boundary layers. For flight at high altitudes, the second mode can be of importance at much lower Mach numbers.

The supersonic stability theory was placed on a firm footing by the experiments of Kendall.[29] Artificial oblique periodic disturbances introduced into a flat-plate boundary layer in a wind tunnel with laminar wall boundary layers at $M_1 = 4.5$ verified the predicted unstable frequencies, amplification rates and phase velocities of the first mode. Further, the existence of amplified second-mode disturbances in the predicted frequency range was confirmed by introducing two-dimensional disturbances. Evidence of the importance of the second mode at hypersonic Mach numbers was later obtained by Kendall[32] in a moderately cooled boundary layer at $M_1 = 7.7$. The measured most unstable frequency in a "naturally excited" boundary layer with turbulent tunnel-wall boundary layers agrees with the calculated most unstable second-mode frequency. Indeed wind tunnel observations in pretransition hypersonic boundary layers of periodic rope-like structures by Potter and Whitfield[33] at $M_1 = 8$, Fischer and Wagner[34] at $M_1 = 14$, Fischer and Weinstein[35] also at $M_1 = 14$*‡ and Demetriades[37] at $M_1 = 7$ (the last three with pictures) support the idea of second-mode instability as a dominating factor. The measured wave lengths in all of these experiments are about 2δ, which is close to the wave length of the most unstable second-mode disturbances. To the objection that these structures are almost certainly nonlinear, it may be recalled that in the low-speed experiments of Klebanoff, Tidstrom, and Sargent[38] with large amplitude artificial disturbances, not only did the original frequency persist to the transition region with little harmonic buildup, but the phase velocity given by the linear theory for that particular frequency was also preserved almost to the point of final breakdown. In a flow without an artificial disturbance, the most likely source of a dominant frequency is selective amplification of linear instability waves with a similar persistence of the frequency of the most-amplified disturbance into the nonlinear range.

III. Numerical Techniques

The numerical results in this paper were obtained from the author's Fortran program VSTAB. This program, an early version of which is described in Ref. 24, solves the eigenvalue problem of linear viscous stability theory for two- or three-dimensional sinusoidal disturbances with either the frequency ω complex (temporal theory), or the wave number α complex (spatial theory), or both complex (temporal/spatial theory). The mean boundary layer may be two- or three-dimensional. An eighth-order system of ordinary differential equations is solved for three-dimensional disturbances or a three-dimensional boundary layer; a sixth-order system for two-dimensional disturbances in a two-dimensional boundary layer.

A forward integration method is used to integrate the analytic independent solutions in the freestream from some arbitrary point well out in the boundary layer to the wall. Most of the calculations of this paper were obtained using a fourth-order fixed step size Runge-Kutta integrator. Both fixed and variable

‡ See a further discussion by Fischer[36] for the actual frequency and a comparison with theory. The less clear wave-like disturbances of Ref. 36 at $M_1 = 7.6$ with wave lengths of 3δ could be related to either first- or second-mode disturbances.

step size Adams-Moulton integrators are available in the program, and the former was used for the calculations of Ref. 25. Although the integrators use double-precision arithmetic, the problem of rapid error buildup is circumvented mainly by the use of Gram-Schmidt orthonormalization, which is applied whenever any of the solutions exceeds a preset value (usually 10^{10}). With this procedure, eigenvalues, and eigenfunctions have been computed with solution growths exceeding 10^{200}. When the wall values of the independent solutions have been obtained, a linear combination of the independent solutions satisfies all but one of the wall boundary conditions. A linear search procedure is used to find a combination of α, ω, and R which satisfies the remaining boundary condition.

The amplitude ratio is the most useful quantity that stability theory provides for application to the transition problem. In the spatial theory any fluctuating quantity, say q, is given by

$$q(x, y, z, t) = Q(y) \exp\left[i(\alpha x + \beta z - \omega t)\right] \quad (2)$$

where $\alpha = \alpha_r + i\alpha_i$, ω is real and both are dimensionless with respect to L^*, the Blasius length scale, and $U_1{}^*$, the velocity scale. The logarithmic derivative of the amplitude of q is

$$(d/dx) \log |q| = -\alpha_i \quad (3)$$

which identifies $-\alpha_i$ as the spatial amplification rate. In terms of $R = Re^{1/2}$ and introducing A for $|q|$,

$$\ln(A/A_o) = -2\int_{R_o}^{R} \alpha_i \, dR \quad (4)$$

In Eq. (4), A_o is the amplitude at R_o, and the integration is carried out with the dimensional frequency $\omega^* = 2\pi f^*$ const. Since the dimensionless frequency $\omega = \omega^* x^*/U_1{}^* R$ varies with x^* for const ω^*, the dimensionless frequency

$$F = \omega^* v_1{}^*/U_1{}^{*2} = \omega/R \quad (5)$$

is used instead.

The spatial amplification rate $-\alpha_i$ is computed as a function of R for fixed F and Eq. (4) is used to obtain A/A_o. With R_o chosen to be the neutral-stability point first encountered by the disturbance, A_o is the amplitude at the start of the unstable region. It is determined by the interaction of the disturbance source with the stable portion of the boundary layer, and its evaluation is a major part of any stability-based transition theory. The simplest situation is where A_o is independent of F and R so that the dependence of A on F and R will be the same as that of A/A_o.

A few remarks are necessary concerning the procedures followed to produce the numerical results to be presented in the next section. The temporal theory was used at Mach numbers above 3 to obtain curves of $\alpha c_i/c_r$ vs R for const F, and this quantity was corrected approximately to the spatial amplification rate by multiplying by the ratio c_r/c_g, where c_g is the group velocity $\partial\omega/\partial\alpha$ computed at the maximum value of $\alpha c_i/c_r$ for each curve. Most of these results are taken from Ref. 25. All of the results at $M_1 = 3$ and below were obtained directly from the spatial theory. Values of F and R were selected, a phase velocity c_p and amplification rate $-\alpha_i$ estimated, and the search procedure operated to yield a new $\alpha = \alpha_r + i\alpha_i$. Only a single perturbation integration is required per iteration for the complex number α.

About 35 eigenvalues are needed for each boundary layer to cover the region up to $R = 2500$ with reasonable completeness. The first few eigenvalues for each boundary layer usually required two iterations, but subsequently one iteration sufficed in most cases, since with the aid of two graphs, one of c_p vs ω and the other of α_i vs R, it was possible to make good initial estimates. With fourth-order interpolation from the mean boundary-layer tables, the Runge-Kutta integrator, Gram-Schmidt orthonormalization whenever the largest solution reached 10^{10} and 100 integration steps (not the optimum, but a safe number), VSTAB requires about 6 sec on a Univac 1108 computer to perform a single integration of the four independent solutions of the eighth-order system. Unfortunately, this integration time is sufficiently large to make it infeasible to completely automate the calculation. The eigenvalues were computed separately at as wide intervals as possible, and a hand-drawn curve of α_i vs R used to calculate A/A_o vs R in a separate computation.

IV. Application of Stability Theory to Transition

Effect of Mach Number

The difficulty of reconciling transition measurements in different wind tunnels is well known. In order to have consistent results against which to test the stability theory, the most desirable transition measurements are those made on a single model in a single wind tunnel with high-quality flow, and in which a large amount of information is available about the disturbance environment. As mentioned in the Appendix, the relation between flat plate and cone transition measurements is uncertain. Since the stability theory applies strictly only to flat plate boundary layers, flat-plate transition data are to be preferred. Unfortunately, lack of sufficient flat-plate data will make recourse to cone data unavoidable.

The transition measurements of Coles[39] on a flat plate in the JPL 20-in. wind tunnel at four Mach numbers meet all of the requirements and are given in Fig. 1 for $Re/\text{in.} = 3 \times 10^5$. Of Coles' four transition criteria, one of the two start-of-transition criteria is naturally to be preferred for comparison with stability theory, and the minimum static pressure point has been selected because there are many more data points than for the minimum shearing stress. The Mach number range covered by Coles is only from 2.0 to 4.5, and other data must be used for other Mach numbers. Two start-of-transition Re_t measured by Deem and Murphy[40] in two different wind tunnels at $M_1 = 5$ and 8 and based upon the minimum pitot-pressure criterion are also shown in Fig. 1 for the same unit Reynolds number. This increase in Re_t with increasing M_1 for $M_1 > 5$ is the one apparently universal feature of all wind-tunnel transition data, and is found for both hollow-cylinder models[11] and cones[41] as well as flat plates.

For $M_1 < 2$ only cone data are available. Laufer and Marte[42] measured transition Reynolds number based on the peak in the surface temperature distribution, which is an end-of-transition criterion, in the JPL 12- and 20-in. wind tunnels. Four measurements in the 20-in. tunnel at $M_1 = 1.3$, 1.45, and 1.8 all gave an Re_t of about 6.2×10^6. Two measurements in the JPL 12-in. tunnel at $M_1 = 1.6$ and 1.8 gave 7.1×10^6 for the former and 6.2×10^6 for the latter. Since the 12-in. tunnel data rather surprisingly agree exactly with those of the 20-in. tunnel for $M_1 \geqq 2$, it is tempting to consider that Re_t may reach a maximum at $M_1 = 1.6$. In any case, judging by the Laufer-Marte result that the cone Re_t at $M_1 = 2$ is about 1×10^6 greater than Coles' maximum static pressure Re_t, which in turn

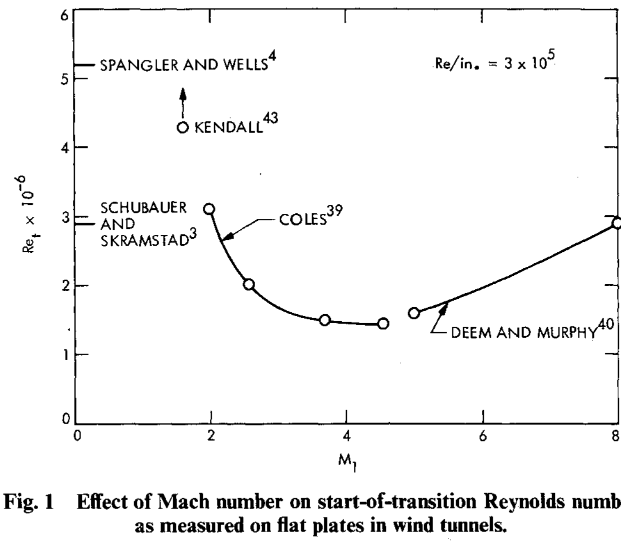

Fig. 1 Effect of Mach number on start-of-transition Reynolds number as measured on flat plates in wind tunnels.

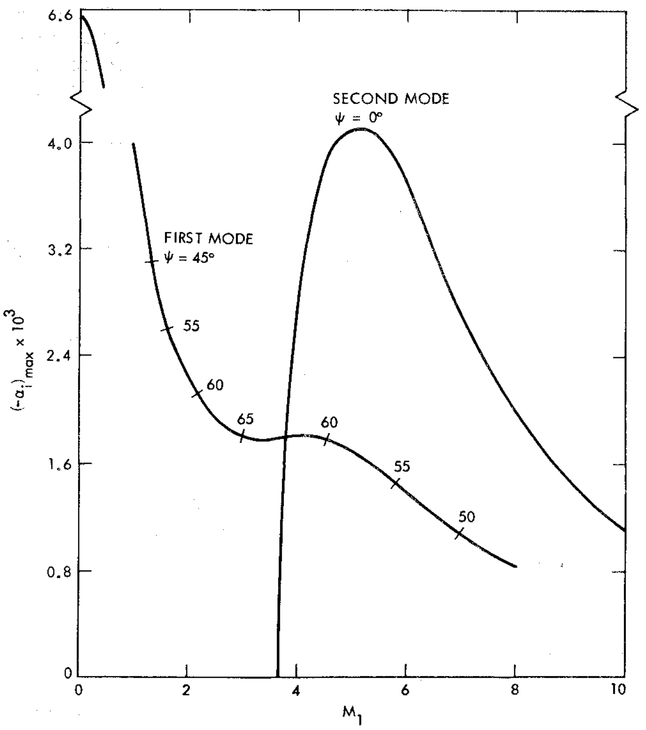

Fig. 2 Effect of Mach number on maximum spatial amplification rate at $R = 1500$. Insulated wall.

is about 1×10^6 greater than the minimum static pressure Re_t, a flat plate start-of-transition Re_t at $M_1 = 1.6$ of 4.2–5.0×10^6 is suggested. The only flat-plate result in the 20-in. tunnel at $M_1 = 1.6$ is by Kendall,[43] who observed no transition at all on a short flat plate with a length Reynolds number of 4.3×10^6 at Re/in. $= 3.4 \times 10^5$. This point is shown in Fig. 1. In view of the uncertainty it is not possible to draw a low Mach number curve in Fig. 1, but the evidence supports a sharp rise in Re_t with decreasing Mach number for $M_1 < 2$. A leveling off for $M_1 < 1.8$, or possibly a maximum near $M_1 = 1.6$, is then indicated. To complete Fig. 1, the flat-plate transition Reynolds numbers at $M_1 = 0$ of both Schubauer-Skramstad[3] and Spangler-Wells[4] are shown.

The easiest quantity to compute in the stability theory is the amplification rate. The variation of the maximum spatial amplification rate at $R = 1500$ with M_1 is given in Fig. 2 for both the first and second modes. On the basis of this figure alone, the disagreement of stability theory with wind-tunnel measurements of Re_t would have to be judged profound. In the Mach number range from 1.6 to 4, where Re_t decreases at least a factor of 3, $(-\alpha_i)_{max}$ actually decreases about 40%. Only for $M_1 > 6$ is the increasing stability of both the first and second modes in agreement with the trend of increasing transition Reynolds number.

Since the amplification rate alone is of little help in studying transition, it is necessary to turn to the growth of single-frequency disturbances as measured by the amplitude ratio A/A_o. A series of growth curves for several frequencies which have important growth in the Reynolds number range up to 2500 have been prepared for insulated-wall boundary layers at $M_1 =$ 1.3, 1.6, 2.2, 3.0, 4.5, and 5.8, and for cooled-wall boundary layers at $M_1 = 3.0$ with $T_w/T_{aw} = 0.90$ and 0.80. These growth curves are given in Fig. 3. The wave angle chosen for the insulated-wall boundary layers is the angle of maximum amplification rate at $R = 1500$. It is not necessarily the angle of maximum amplitude ratio, but should be close to it. For the cooled-wall boundary layers at $M_1 = 3$, the growth curves are for $\psi = 55°$ rather than the most unstable angle of 65°, and there is also an insulated-wall growth curve for the same angle. These three curves were prepared for use with the forcing theory to be presented in Sec. V. For $M_1 = 4.5$, curves of individual frequencies are not given, but only the envelope curve of maximum growth with the frequencies indicated along the curve.

The growth curves of Fig. 3 provide the material for the testing of stability theory as an indicator of transition trends. The most direct method of applying the curves is by means of an amplitude-ratio criterion as was done by Smith et al.[15,17] for incompressible flow. With the e^9 criterion it is estimated that the Reynolds number at which this amplitude ratio is reached is approximately 20×10^6 at $M_1 = 2$ and 10×10^6 at $M_1 = 4.5$. Such high transition Reynolds numbers have been observed in flight, and it may be noted that at $M_1 = 4.5$ the laminar boundary layer on the wall of the JPL 20-in. tunnel in the test section at Re/in. $= 0.45 \times 10^5$ has a thickness of 1.7 in.[43] The equivalent flat plate x-Reynolds number is 27×10^6, although the velocity profile bears no resemblance to the flat-plate profile. In order to obtain values of Re more closely related to observed transition Reynolds numbers in supersonic wind tunnels operating with turbulent boundary layers on the walls, an arbitrary choice of $A/A_o = 100$ and 400 has been made. The intersection of the envelope or maximum growth curve (extrapolated if necessary) with the selected level of A/A_o gives an R whose square will be designated Re_A. The interest will lie in whether or not the variation of Re_A with freestream Mach number bears any resemblance to Re_t vs M_1 as exemplified by Fig. 1. The actual magnitude of Re_A is of secondary importance. The two curves marked $A_o = A_r$ in Fig. 4 are the results.§ A_r is a reference amplitude which is constant in this calculation. Certain features of the Re_t curve are indeed indicated, primarily a minimum near $M_1 = 4$. The ratio of $(Re_A)_{max}$ to $(Re_A)_{min}$ in Fig. 4 decreases with decreasing A/A_o but is always greater than unity. There has long been an opinion that the "bucket" in Fig. 1 is strictly a wind-tunnel phenomenon. However, Fig. 4 demonstrates that on the present approach this feature is a direct consequence of the amplitude growth curves of Fig. 3. If the minimum is indeed related to stability theory, and stability theory determines the variation of Re_t with M_1, then it should be observed elsewhere than in wind tunnels. The maximum near $M_1 = 2$ may or may not be in accord with experiment as previously discussed. The two $M_1 = 0$ transition Reynolds numbers of Refs. 3 and 4 are noted in Fig. 4 along with the equivalent A/A_o factors calculated in Ref. 17.

The transition criterion $A/A_o =$ const. can be interpreted either as an amplitude-ratio criterion with A_o variable, or as an amplitude criterion with A_o constant. The first interpretation has little physical meaning and will not be pursued further, but an amplitude criterion, according to which transition will be considered to be triggered whenever the disturbance reaches some critical amplitude independent of Mach number, is a reasonable one for the purposes of this paper. However, the disturbance environment changes with Mach number and A_o must change also. Consequently, the transition criterion is reformulated as

$$A/A_r = (A_o/A_r)(A/A_o) = \text{const} \qquad (6)$$

where A_r is a fixed reference amplitude, and A_o/A_r and A/A_o are both functions of Mach number as well as the other flow parameters which influence the mean boundary layer. A_o/A_r is determined by the amplitude (and spectrum) of the external disturbances and the interaction of these disturbances with the boundary layer; A/A_o is the contribution to the final disturbance amplitude from instability.

It is now well established that transition in supersonic wind tunnels is heavily influenced by the sound radiated from the turbulent boundary layers on the tunnel walls. Measurements of Laufer[44] show that the rms pressure fluctuation in the freestream varies essentially as $M_1{}^2$ from about $M_1 = 1.6$

§ The points at $M_1 = 7$ were obtained from an approximate growth curve for $\psi = 60°$. Both the approximate nature of the calculation and the fact that this angle may not be the most unstable angle (see Fig. 13-12 of Ref. 25) makes it necessary to treat these points with reserve.

to 5 (and particularly from $M_1 = 3$ to 5) for Re/in. $\approx 3.4 \times 10^5$. An elementary way to account for this effect within the framework of Eq. (6) is to let

$$(A_o/A_r) = (M_1/M_r)^2 \quad (7)$$

Thus A_r is identified as A_o at the reference Mach number M_r, and A_o varies with $M_1{}^2$ as desired. The instability amplification required to satisfy the criterion Eq. (6) is inversely proportional to $M_1{}^2$. A number of assumptions are implicit in this procedure

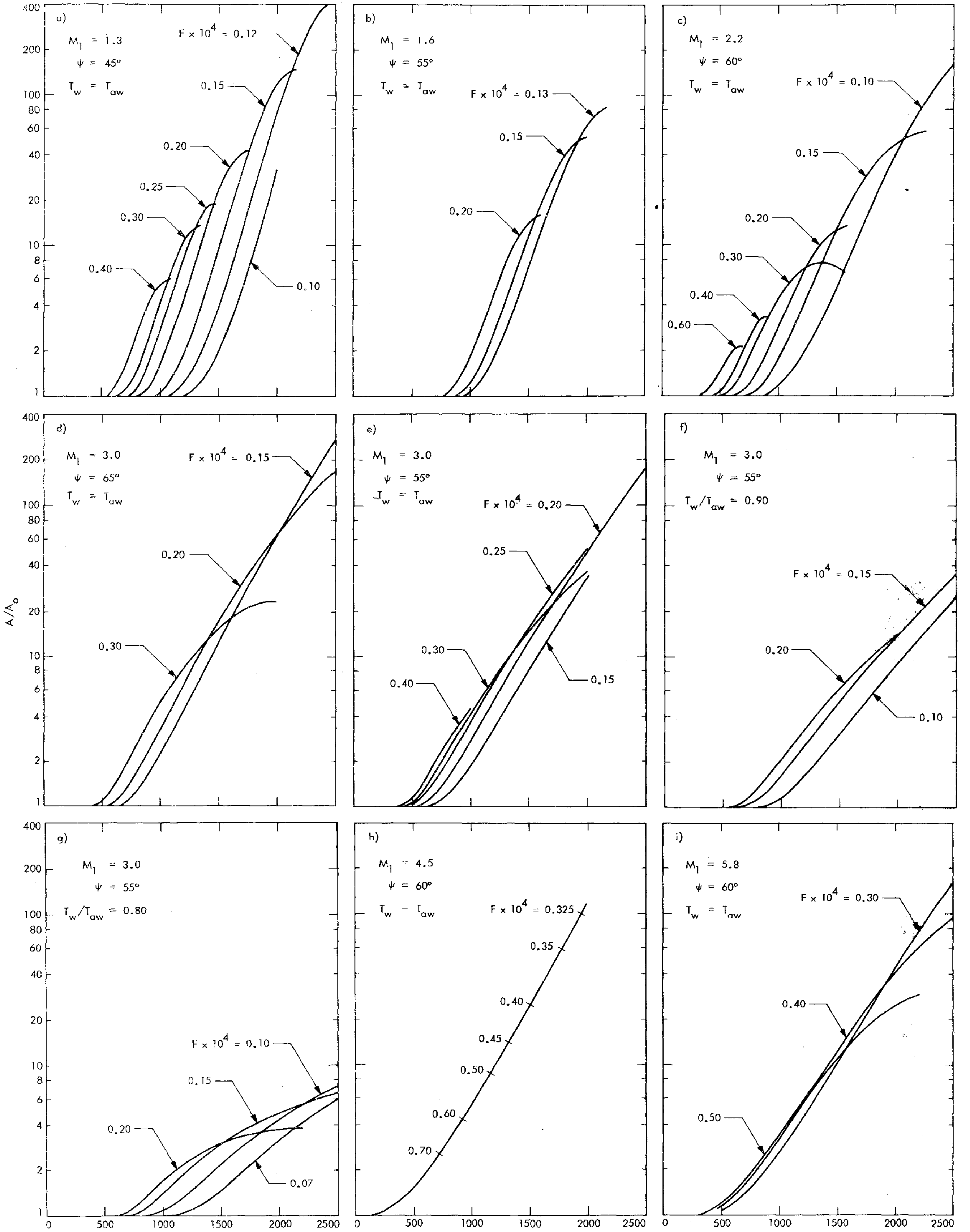

Fig. 3 Amplitude growth curves of single-frequency three-dimensional disturbances from $M_1 = 1.3$ to 5.8.

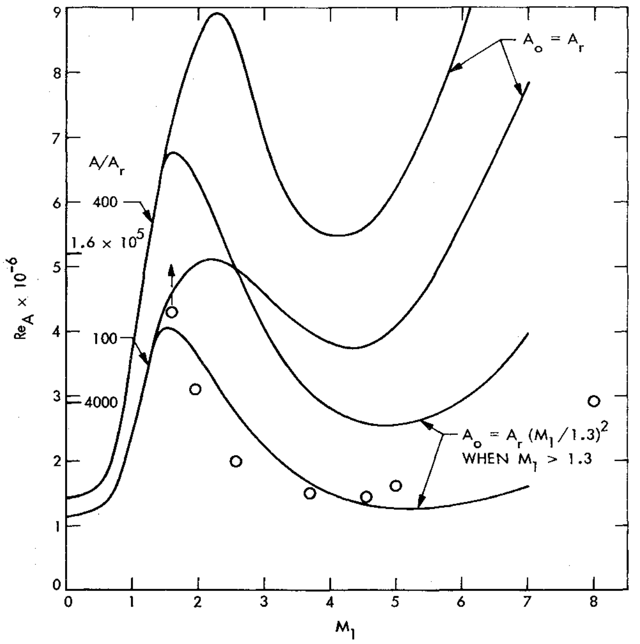

Fig. 4 Theoretical calculations of effect of Mach number on transition for two A/A_r and two assumptions about initial disturbance amplitude. Insulated wall. Experimental points are from Fig. 1 (Re/in. $= 3 \times 10^5$).

and it is well to list them. The primary assumption is that the source of the boundary-layer disturbances which controls transition is the irradiated sound, and that the disturbances in the boundary layer, no matter what the details of the interaction process, are proportional in amplitude to the freestream pressure fluctuations. It is further assumed that the disturbance spectrum in the boundary layer is flat with respect to both frequency and wave angle, and that the amplitude of the initial disturbance is independent of its x-position in the boundary layer.

In the application of Eqs. (6) and (7), M_r is taken to be 1.3 and for $M_1 < 1.3$, A_o is considered to be constant and equal to A_r. The calculation proceeds as follows: with a particular value of A/A_r chosen as the transition criterion, A_o/A_r is obtained at each M_1 from Eq. (7), and then A/A_o is found from Eq. (6). With A/A_o known, the amplitude-growth curves of Fig. 3 are used as before to determine Re_A. The results for $A/A_r = 100$ and 400 are shown in Fig. 4 where the experimental points from Fig. 1 have been repeated for convenience of reference. In both cases, the shape of the curve has been improved from the previous calculation with the maximum moved to below $M_1 = 2$ and the ratio of maximum Re_A to minimum Re_A about right although the minimum perhaps is overshifted to a higher Mach number. Detailed comparisons with the experimental data are not warranted because Re_A, unlike Re_t, is independent of unit Reynolds number.

A unit Reynolds number effect can be included in the present approach by multiplying the right-hand side of Eq. (7) by a ratio of Re/in. raised to some power. The measurements of Laufer[44] suggest that the rms amplitude of the freestream pressure fluctuation amplitude varies as $(Re/\text{in.})^{-0.3}$ for $M_1 > 2.2$. When this factor is used in Eq. (7), it is found that between $Re/\text{in.} = 1 \times 10^5$ and 3×10^5 $Re_A \sim (Re/\text{in.})^n$, where n increases from $n = 0.12$ at $M_1 = 2.2$ to $n = 0.24$ at $M_1 = 5.8$. These values of n are smaller than is usually found with experimental data to which a power law can be fitted.

The main feature of an actual boundary-layer disturbance that has been neglected so far is its spectrum. In a complete calculation with a spectrum, the A_o given by Eq. (7) refers to the average amplitude of all frequencies. The A_o of the individual frequencies are adjusted according to the distribution of A_o, or more precisely the square root of the spectral energy density, in a spectrum normalized to have an average value of unity. It is reasonable to suppose that the spectrum of A_o is related to the spectrum of freestream disturbances, particularly for $M_1 > 2.2$ where Kendall[32] has shown a strong correlation between freestream and boundary-layer disturbances. Kendall[32] has measured one-dimensional spectra at several Mach numbers, but except for $M_1 = 2.4$ they are all at a low Re/in. of about 1×10^5 where there are few transition data. The lack of data for comparison, together with a change in the character of the spectrum at $M_1 = 4.5$ for $Re/\text{in.} < 1.4 \times 10^5$, means that a calculation using spectra at $Re/\text{in.} = 1 \times 10^5$ offers little reliable new information beyond what can already be gathered from Fig. 4. As it stands, Fig. 4 offers a considerable amount of support for the idea that the change of Re_t with Mach number is governed in large part by what happens in the linear instability region.

Effect of Wall Cooling at $M_1 = 3$

The need to account for the influence of a changing disturbance environment on transition can be avoided by concentrating on the effect of wall cooling. With the Mach number and unit Reynolds number constant, the external flow is unchanged as the boundary layer is cooled. As is well known, this apparently simple experimental situation has led to an astonishing variety of observations. A delay of transition by cooling, transition reversal (decrease of Re_t with cooling after an initial increase), transition re-reversal (increase of Re_t after a previous decrease), unexpectedly early transition and no effect of cooling whatever are some of the phenomena which have been found in different types of experimental facilities, in different examples of the same kind of facility, and in the same facility under different flow conditions. For wind tunnels, the situation as summarized by Morkovin[18] is that transition can be delayed by cooling, with the delay most pronounced at low Mach numbers, and that above a certain Mach number, perhaps about $M_1 = 6$ (now raised to at least 6.8 by Ref. 45), there is no longer an effect of cooling. Transition reversal is observed in some wind tunnels but not in others, and where it is observed occurs at different values of the temperature ratio.

The approach here is the same as for the effect of Mach number. First, some reliable experimental data will be collected, then the effect of cooling on the amplification rate will be examined, and finally the ratio of Re_A to $(Re_A)_{aw}$, the Re_A for no cooling, will be calculated as a function of the wall temperature ratio T_w/T_{aw}. The calculation will be done at $M_1 = 3$ which is low enough to preclude any possible influence of the second mode but at the same time high enough to assure that the irradiated sound has a dominant influence. Since no flat-plate data are available, cone data from the JPL 12- and 20-in. wind tunnels are given in Fig. 5 in the form of the ratio of Re_t to $(Re_t)_{aw}$ as a function of T_w/T_{aw}. There are two faired curves in Fig. 5, one from measurements of Van Driest and Blumer[46] on a 10° cone at $M_1 = 2.7$ (edge Mach number) in the 20-in. tunnel, and the other from Van Driest and Boison[47] at the same Mach number and also on a 10° cone, but in the 12-in. tunnel. No transition reversal was observed in either experiment. The transition point was determined optically in both experiments by means of a schlieren system with vertical expansion. In Ref. 47 measurements of the temperature distribution for noncooled cones showed that the optical technique corresponded to the peak in the temperature distribution, and is therefore an end-of-transition criterion. The measurements are presented in Fig. 5 as ratios to help eliminate any influence of using an end-of-transition criterion for comparison with theory instead of a more appropriate start-of-transition criterion, and also to avoid the poorly understood differences between transition Reynolds numbers measured on cones and flat plates.

Figure 6 gives the effect of cooling on the maximum amplification rate of the first mode at three Reynolds numbers for 55° waves at $M_1 = 3.0$. This angle will be used for the cooling calculation in preference to 65° so that the forcing theory of the next section can be applied. If Fig. 3e is compared

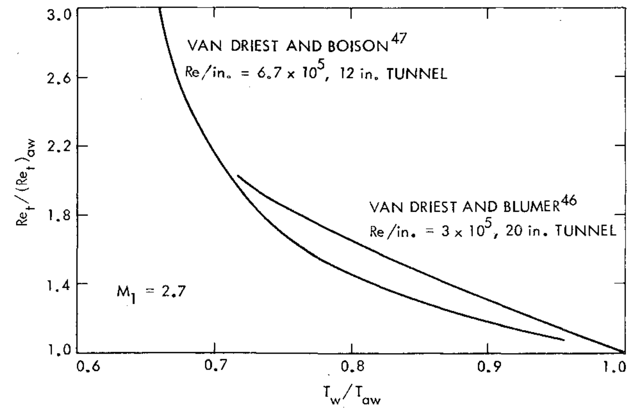

Fig. 5 Measured increase of cone transition Reynolds number with cooling at $M_1 = 2.7$ in two different wind tunnels.

with Fig. 3d, it will be seen that 55° waves are almost as unstable as 65° waves. Figure 6 indicates complete stabilization at about $T_w/T_{aw} = 0.6$. The boundary layer is completely stabilized to inviscid disturbances at a T_w/T_{aw} just below 0.7, and Fig. 6 is in accord with this résult in suggesting that as R decreases more cooling is required for complete stabilization. Disturbances at some finite Reynolds number, perhaps near 1000, will be the last to be stabilized. The comparison of Figs. 5 and 6, in contrast to the comparison of Figs. 1 and 2, shows that the effect of cooling on the transition Reynolds number at Mach numbers near 3 is in complete accord with the behavior of the amplification rate.

The calculation of Re_A from Figs. 3e, 3f, and 3g (extrapolated where necessary) with $A/A_o = 100$ gives a rise in $Re_A/(Re_A)_{aw}$ with decreasing T_w/T_{aw} much more rapid than the observed increase of Re_t in the 20-in. wind tunnel. For $T_w/T_{aw} = 0.7$, Re_A could not be calculated from the available amplitude-growth curve (not shown in Fig. 3), but was estimated to be well over 10^{10}, and complete stabilization would occur by $T_w/T_{aw} = 0.6$. When $A/A_r = 100$ is used in place of A/A_o as the transition criterion, the required A/A_o drops to 18.8 and much lower Re_A are obtained. However, the ratio $Re_A/(Re_A)_{aw}$ is little affected. Since the complete program of desired calculations cannot be carried through with $A/A_r = 100$ because of the limited number of frequencies in Fig. 3, $A/A_r = 50$ will be used instead. Again lower Re_A are obtained than with $A/A_r = 100$, but the ratio $Re_A/(Re_A)_{aw}$ at $T_w/T_{aw} = 0.90$ is unchanged and is only 5% lower at $T_w/T_{aw} = 0.80$. These results are shown in Fig. 7 as the curve labelled "instability." A portion of the Van Driest-Blumer experimental curve from Fig. 5 is repeated in Fig. 7 as a broken line, and it is apparent that the technique that gave good results for the Mach number effect is inadequate in the present instance. Of course, it must be kept in mind, as pointed out in the Appendix, that cone experience may not be transferable to the flat plate.

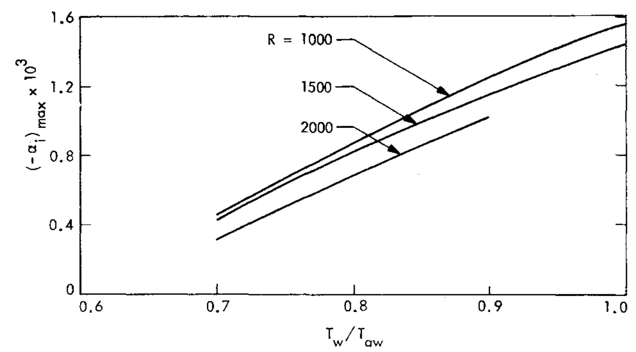

Fig. 6 Effect of cooling on maximum spatial amplification rate at three Reynolds numbers and $M_1 = 3$, $\psi = 55°$.

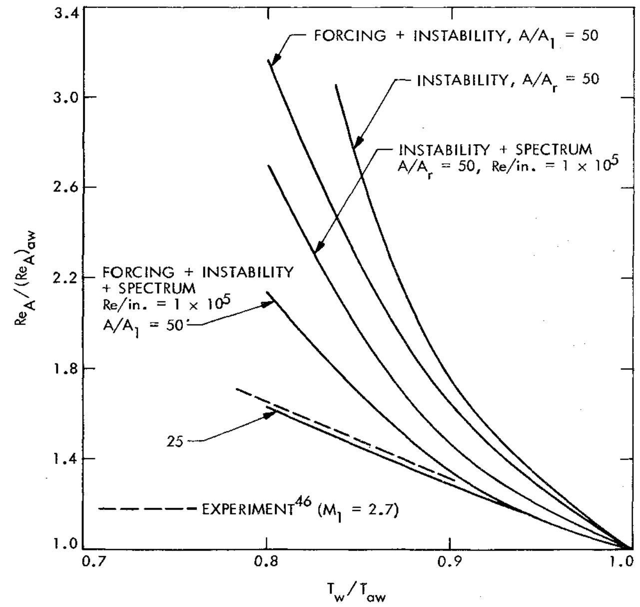

Fig. 7 Five theoretical calculations of the effect of cooling on transition at $M_1 = 3$.

The first improvement that can be made to the preceding calculation is to make use of one of the freestream disturbance spectra measured by Kendall,[32] even though these spectra are one-dimensional and the theory really requires a two-dimensional spectrum (frequency and wave angle). As the boundary layer is cooled, the most unstable frequency for a given A/A_o decreases and thus moves to a part of the spectrum with increased energy. As a result less amplification is required to reach the assigned amplitude. The spectrum measured by Kendall[32] at $M_1 = 3.25$ and $Re/\text{in.} = 1 \times 10^5$ was used in the calculation. A considerable shift in the shape of the spectrum at the relevant frequencies occurs at $M_1 = 2.4$ as the Re/in. increases from 1×10^5 to 3×10^5, but it is not known if a similar shift occurs at $M_1 = 3$. There is no documented effect of unit Reynolds number on $Re_t/(Re_t)_{aw}$.

The calculation proceeds by adjusting the curves of individual frequencies in Fig. 3 upward or downward by the square root of the energy density of the power spectrum normalized to have unit area. The final result is the curve in Fig. 7 labelled "instability + spectrum." There is seen to be a definite improvement over the previous calculation, but the cooling effect is still larger than observed experimentally. To progress further it is necessary to give more attention to the question of how A_o is related to the freestream disturbances. The theory of the next section is designed to answer this question, and the remaining curves of Fig. 7 will be discussed after that theory is presented.

V. Forced Response of Boundary Layer

Outline of Theory

The structure of linear stability theory allows the forced response of the boundary layer on a flat plate to a particular type of external disturbance field to be readily obtained. One of the independent solutions of the stability equations is, for $\alpha_i = 0$ and in the limit of large Reynolds number, the inviscid flow over an oblique wavy wall of wave length $2\pi/\alpha$ moving with phase velocity c_r. The pressure fluctuation given by this solution is, apart from the time factor,

$$p' = i\alpha\gamma M_1{}^2(1-c_r)\exp\{i\alpha[x \mp (\hat{M}_1{}^2-1)^{1/2}y]\} \qquad (8)$$

For a disturbance which is oblique to the freestream, α and $\hat{M}_1$ are taken in the direction normal to the constant-phase line in the x-z plane. It is seen from Eq. (8) that when $\hat{M}_1 > 1$ the constant-phase lines in the x-y plane are Mach waves. With the negative sign in Eq. (8), the Mach waves are outgoing, i.e., energy is transported in the direction of increasing y; with the positive sign, the Mach waves are incoming. When $\hat{M}_1 < 1$, the solution with the upper sign decays exponentially upward, and the other solution increases exponentially upward. In stability theory, only solutions which are at least bounded as $y \to \infty$ are permitted, but no such restriction is present in the forcing theory where the incoming disturbance has been produced elsewhere in the flow. The full viscous counterpart of Eq. (8) for an incoming wave has a slow exponential increase upward which is perfectly acceptable.

The incoming-wave solution bears some resemblance to a Fourier component of the sound field radiated from turbulent boundary layers at high supersonic speeds according to Phillips' theory.[48] In this theory, each Fourier component (α, β) is produced by the same Fourier component of the frozen turbulent field moving at a supersonic source velocity c_r relative to the freestream. Thus the turbulent boundary layer is decomposed into oblique wavy walls moving supersonically, and the associated outgoing Mach waves are the incoming Mach waves of the receiving laminar boundary layer at $y = 0$. However, in Phillips' theory the field is random and each "wavy wall" exists for only a finite time related to the lifetime of an individual eddy. In the present theory, the incoming disturbance field is steady to an observer moving with c_r.

A solution for the boundary-layer response at each Reynolds number can be found for each α, β (or ψ), and c_r by using both inviscid solutions of the eighth-order system of equations of linear stability theory together with the usual three viscous solutions which go to zero as $y \to \infty$ to satisfy the boundary conditions at $y = 0$. The combined solution, in addition to giving the boundary-layer disturbance which results from the external acoustic disturbance, also provides the amplitude and phase of the outgoing, or reflected, wave relative to the incoming wave. The response disturbance is neutral in the sense of stability theory, but its amplitude in the boundary layer is a function of Reynolds number. If the local mass-flow fluctuation amplitude m is chosen to represent the amplitude (a hot-wire anemometer measures primarily m), the ratio of m_p, the peak value of m, to m_I, the mass-flow fluctuation of the incoming wave, can be called A/A_1 and used in a manner similar to the amplitude ratio A/A_o of an instability disturbance. An increase in m_p/m_I with increasing R represents an "amplification"; a decrease, a "damping."

The stability program VSTAB has an option which permits the boundary-layer response to an incoming wave to be computed. With values of F, R, and c_r specified, the program performs two integration passes through the boundary layer. In the first pass, the five independent solutions are integrated from the freestream to the wall, and a linear combination determined which satisfies the boundary conditions. The integration is repeated with the known combining coefficients to provide m/m_I as a function of y/δ.

Some numerical results from the forcing theory may be found in Ref. 49. The main result of the theory, which has been found to be true for all boundary layers and all disturbances regardless of wave angle and phase velocity (provided only that $\hat{M}_1 > 1$), is that the amplitude ratio m_p/m_I starts to grow at $x = 0$, reaches a peak whose x distance and magnitude are inversely proportional to the frequency, and then decreases slowly. The decrease is given closely by the inviscid theory, but the increase is a viscous phenomenon. As a consequence of this behavior, the forcing mechanism provides boundary-layer disturbances from 5–20 times as large as the freestream disturbances without any instability amplification.

A difficulty exists concerning the proper value of c_r to use. Other than the requirement that $c_r < 1 - 1/M_1 \cos \psi$ to insure a supersonic relative flow, there is no theoretical guidance and recourse must be had to experiment. Space-time correlation measurements by Laufer[44] and Kendall[29] in the freestream gave an average source velocity for the entire hot-wire signal. Unfortunately, there is a difference of about 20% in the two measurements. Kendall also measured a dispersion of about 40% at $M_1 = 4.5$. The redeeming feature is that the calculations show c_r to have only a small influence on the results.

Combination of Forcing and Stability Theories

The quantity m_p/m_I, interpreted as A/A_1, is the most important result of the forcing theory for application to the transition problem. It provides the essential piece of information which has been missing up to now: the relation of the amplitude of a boundary-layer disturbance to the amplitude of a freestream disturbance. Strictly speaking, m_p is equivalent to the A of stability theory only when the m distribution through the boundary layer is self-similar, but such is not always the case. However, this situation is no different from what is involved in comparisons of stability theory with experiments in which a peak m is followed downstream and identified with A even when nonsimilar amplitude distributions are involved.

The major difficulty in the use of the forcing theory is that forced disturbances are distinct from free disturbances, and the process by which the former become the latter is unknown. An experiment by Kendall[50] showed that, as measured by the phase velocity, a forced disturbance near the leading edge evolves into a free disturbance farther downstream. It will be *assumed* here that the forcing theory applies up to the neutral-stability point of the particular frequency under consideration, and that the stability theory applies downstream of that point. The conversion from one disturbance to the other would seem most likely to occur if the amplitude distribution of the forced disturbance through the boundary layer at the neutral-stability point matched the eigenfunction. A limited number of calculations at $M_1 = 4.5$ show that the two distributions are indeed close together for the same α and R. With the only mismatch between the two disturbances a phase velocity difference of 20%, conversion of forced into free disturbances can be expected to take place quickly.

Consequently, with the approach just outlined the forcing theory is used primarily to calculate A_o/A_1, the disturbance amplitude ratio at the neutral point. The subsequent amplitude ratio of the disturbance is found by multiplying A_o/A_1 by A/A_o calculated from stability theory. With the stability theory alone and A_o either constant or proportional to $M_1{}^2$, all frequencies have the same initial amplitude. On the contrary, with the forcing theory the interaction of the external acoustic disturbances with the boundary layer is frequency dependent and A_o is a function of frequency.

Application to Effect of Wall Cooling

The first task in the use of the combined theory to continue the study of the effect of wall cooling is to prepare amplitude growth curves for A/A_1 from the two theories. An example of a combined curve is shown in Fig. 8 for $T_w/T_{aw} = 0.8$ and $\psi = 55°$. The phase velocity in the forced region is 0.40 (Laufer[44]), and $\psi = 55°$ was chosen because it satisfies the requirement $\hat{M}_1 > 1$ and is also close to the most unstable wave angle. In Fig. 8, the neutral points, which are the minima denoted by tick marks, occur well after the maxima reached in the forced region. For $0.8 < T_w/T_{aw} < 1$ at $M_1 = 3$, there is almost no effect of temperature on the forced response for 55° waves. The transition criterion is now A/A_1 rather than Eq. (6). Since $A_o/A_r = 5.33$ at $M_1 = 3$ from Eq. (7), and $A_o/A_1 \approx 5$ according to the forcing theory for the relevant frequencies, values of A/A_1 equal to A/A_r will lead to about the same Re_A. The curve in Fig. 7 labelled "forcing + instability" is the result of calculations using the combined growth curves and a transition criterion of $A/A_1 = 50$. The ratio $Re_A/(Re_A)_{aw}$ is reduced by 30% at $T_w/T_{aw} = 0.8$ from the value obtained with stability theory alone, but is still well above the experimental curve.

A final calculation combines all the elements of the previous ones: the forcing theory, the stability theory, and the same spectrum of freestream disturbances as was used in Sec. IV. The result for $A/A_1 = 50$ is the curve labelled "forcing + instability + spectrum" in Fig. 7. There is a reduced effect of

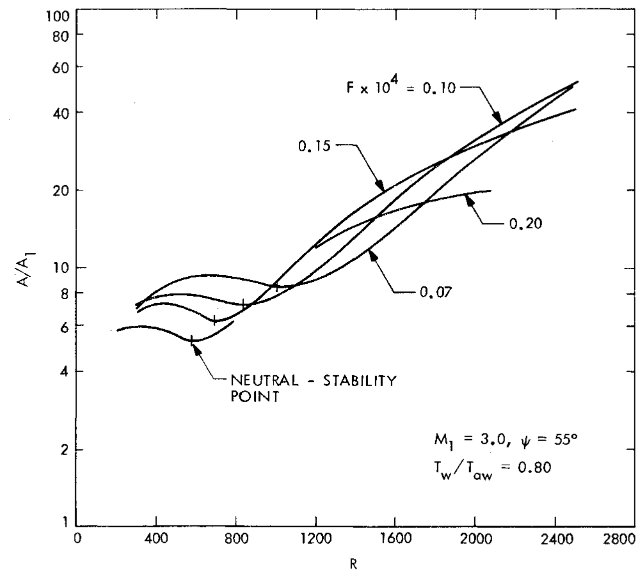

Fig. 8 Amplitude growth curves of single-frequency three-dimensional disturbances from combined forcing and stability theories at $M_1 = 3$, $T_w/T_{aw} = 0.8$, $\psi = 55°$.

cooling compared to the other calculations, but the percent increase in Re_A at $T_w/T_{aw} = 0.8$ is almost double the experimental increase. In order to determine the sensitivity of the result to the particular A/A_1 used, the calculation was repeated with $A/A_1 = 25$. As can be seen in Fig. 7, there is a sizable effect of A/A_1, and the new curve happens to almost duplicate the Van Driest-Blumer curve. It must be emphasized, however, that the experimental curve refers to a cone (see Appendix), is for a slightly different Mach number and much different unit Reynolds number, and uses an end rather than a start-of-transition criterion.

For $T_w/T_{aw} = 0.7$, instability amplification is almost nil and even with the forced response an A/A_1 of 25 is not reached until a very high Reynolds number. For $T_w/T_{aw} = 0.6$, only the forced response is left. Since transition certainly still occurs, the question remains as to the cause. It could be the forced response to the irradiated sound, or it could be some other unknown cause that is normally unimportant as long as instability amplification is present. If the former, then a new transition criterion, perhaps involving the Reynolds stress, must be sought as even an A/A_1 of 25 is never achieved by the forced response according to the present theory. The present results suggest that one property of a more realistic criterion would be that the disturbance amplitude at transition is a decreasing function of Reynolds number. Of course, the forcing theory used here is a simplified one and the actual forced response to a random sound field generated by turbulent eddies of short lifetime could be quite different. In any event, the cause of transition must change somewhere in the neighborhood of $T_w/T_{aw} = 0.8$ from instability amplification with its concentration of disturbance energy in a specific frequency band to a direct interaction of disturbances with the boundary layer. In such a circumstance it would not be surprising to find a different variation of transition Reynolds number with cooling than that prevailing during instability amplification. Also, different disturbance sources could result in different cooling effects. The multitude of phenomena observed at the lower values of T_w/T_{aw} might be explainable on this basis. For hypersonic Mach numbers, the unstable second mode can provide instability amplification over the whole temperature range.

VI. Conclusions

The aim of this investigation was to determine if observed *changes* in the transition Reynolds number in a supersonic wind tunnel as the freestream Mach number and wall temperature change could be quantitatively accounted for by an elementary use of linear stability theory. The stability theory was applied in the form of maximum amplitude growth curves of three-dimensional disturbances. It proved necessary in both instances to bring in some information about the external disturbances responsible for transition. The stability theory alone is inadequate, because what must be determined is not what can happen to a broad class of disturbances, but what does happen to the particular disturbances which exist in a given flow situation. In this respect the effect of Mach number proved the easier of the two problems. The influence of the irradiated sound field in a supersonic wind tunnel is so strong that it was only necessary to let the initial disturbance amplitude in the boundary layer increase with M_1^2 in accordance with the behavior of the acoustic disturbances to arrive at a reasonable approximation of the variation of transition Reynolds number with Mach number.

The second problem is the more difficult precisely because the disturbance environment is fixed. As the boundary layer is cooled and the instability amplification gradually reduced to zero, transition will evidently depend critically on the exact nature of the external disturbances and the details of their interaction with the boundary layer. By calling upon a linear interaction theory and a measured spectrum of the acoustic disturbances, agreement was finally obtained with the only experimental data with which a comparison could be made. However, the conditions of the data differ enough from those of the theory so that a definitive conclusion cannot be drawn. It is felt, though, that sufficient evidence has been presented to make the premise of this study an acceptable one: the observed variation of the transition Reynolds number with mean-flow parameters and the external disturbance environment depends to a great extent on what takes place in the region before transition where linear theory is valid.

What is needed for further progress is not only the extension of linear theory to more complicated flow situations, but also more patient accumulation of *complete* transition data such as has been accomplished by the NASA Transition Study Group. With such data as exist now, and as more become available, linear stability theory, by further calculations of the type presented here, can aid in bringing order to the present confused subject of supersonic boundary-layer transition.

Appendix: Relation between Stability Theory on a Cone and Flat Plate

Both a flat plate and a cone at zero angle of attack provide a zero pressure-gradient two-dimensional boundary layer. The two laminar boundary layers are related, as is well known, by a factor of three in the length scale. That is, with identical edge conditions and $x_c^* = 3x_f^*$, the mean boundary-layer thickness and the velocity and temperature profiles are identical in the two cases. The only theoretical work on the equivalent transformation of the linearized stability equations is to be found in the short note of Battin and Lin[51] for incompressible flow. The analysis referred to in Ref. 51 is presumably based upon asymptotic theory, and reaches the conclusion that when the boundary-layer thicknesses are equal the amplification rates are also equal. Thus, if $x_c^* = 3x_f^*$

$$(\alpha_i^*)_c = (\alpha_i^*)_f \quad \text{(A1)}$$

The dimensionless α based on δ are also equal, and with the Blasius length scale L^*,

$$\alpha_c = 3^{1/2}\alpha_f \quad \text{(A2)}$$

With the same amplification rate acting over three times the distance on a cone as compared to a flat plate, the conclusion of Battin and Lin[51] follows immediately that

$$(A/A_o)_c = (A/A_o)_f^{\,3} \quad \text{(A3)}$$

where $(Re_A)_c$ and $(Re_o)_c$ are both three times the respective flat-plate values.

On this basis, the amplitude growth curves of Fig. 3 would be replaced for a cone by curves in which the R coordinate is increased by a factor of $3^{1/2}$ and the ordinate $\ln(A/A_o)$ by a factor of 3. Only an infinite slope could give the result that $(Re_A)_c = 3(Re_A)_f$. Furthermore, it would be entirely possible, and even probable, to have $(Re_A)_c < (Re_A)_f$. The problem with the foregoing analysis is that Eq. (A1) has no theoretical justification for compressible flow. For it to be true the complete linearized stability equations, not a simplified version according to order-of-magnitude arguments, would have to transform as do the mean boundary-layer equations.

The most complete experimental study of the relation between cone and flat-plate transition Reynolds numbers has been carried out by Pate.[41] In this work, cone end-of-transition data are compared with hollow-cylinder and flat-plate data from several wind tunnels in the Mach number range from 3 to 8. The ratio $(Re_t)_c/(Re_t)_f$ was found to be about 2.5 near $M_1 = 3$ and to decrease slowly to near unity at $M_1 = 8$. On the other hand, Laufer and Marte[42] measured $(Re_t)_c$ to be no more than 20% above the four $(Re_t)_f$ measured by Coles[39] in the same wind tunnel, but even here $(Re_t)_c$ was in no instance less than $(Re_t)_f$. It is difficult to reconcile either of these results with Eqs. (A1) and (A3) *if* A_o was the same for both cone and flat plate, and *if* the laminar boundary layers of the experiments were indeed the theoretically predicted boundary layers. With both of these conditions met, the only way it appears possible for $(Re_t)_c$ to be larger than, or even equal to, an $(Re_t)_f$ far removed from the critical Reynolds number is for $(\alpha_i^*)_c$ to be substantially smaller than $(\alpha_i^*)_f$ for equal boundary-layer thicknesses.

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